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Departamento de Mecánica de Medios Continuos y Teoría de Estructuras
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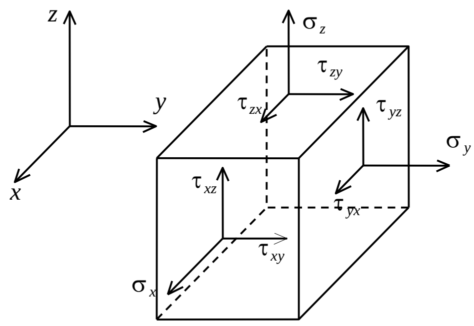
Universidad Politécnica de Madrid

FORMULARIO DE ELASTICIDAD

ELASTICIDAD TRIDIMENSIONAL. COORDENADAS CARTESIANAS



Tensor de deformaciones: $\mathbf{D} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}$

Tensor de tensiones: $\mathbf{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$

En ejes (x', y', z'): $\mathbf{T}' = \mathbf{C}^T \mathbf{T}\, \mathbf{C}$, $\mathbf{D}' = \mathbf{C}^T \mathbf{D}\, \mathbf{C}$
C matriz de cambio de base entre bases ortonormales

Equilibrio interno (tensiones):

$$\sigma_{ij,i} + f_j = 0$$

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{aligned}\right\}$$

Equilibrio interno (movimientos):

$$G u_{j,kk} + (\lambda + G) u_{k,kj} + f_j = 0$$

$$\left.\begin{aligned} G\nabla^2 u + (\lambda + G)\frac{\partial e}{\partial x} + f_x = 0 \\ G\nabla^2 v + (\lambda + G)\frac{\partial e}{\partial y} + f_y = 0 \\ G\nabla^2 w + (\lambda + G)\frac{\partial e}{\partial z} + f_z = 0 \end{aligned}\right\}$$

Equilibrio en el contorno: $\mathbf{T}.\mathbf{n} = (f_x, f_y, f_z)^T$ (**n**, versor normal al contorno)

Compatibilidad en deformaciones:

$$\varepsilon_{ii,jj} + \varepsilon_{jj,ii} = 2\varepsilon_{ij,ij} \text{ (no sumatorio)}$$

$$\left.\begin{aligned} \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial z} \end{aligned}\right\}$$

$$\varepsilon_{ii,jk} = \left(-\varepsilon_{jk,i} + \varepsilon_{ik,j} + \varepsilon_{ij,k}\right)_{,i} \text{ (no sumatorio)}$$

$$\left.\begin{aligned} 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\ 2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\ 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \end{aligned}\right\}$$

Compatibilidad en tensiones (Beltrami – Mitchell): $\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} s_{,ij} = -f_{i,j} - f_{j,i} - \frac{\nu}{1-\nu} f_{\alpha,\alpha}\delta_{ij}$

$$\left.\begin{aligned} \nabla^2 \sigma_x + \frac{1}{1+\nu}\frac{\partial^2 s}{\partial x^2} = -2\frac{\partial f_x}{\partial x} - \frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) \\ \nabla^2 \sigma_y + \frac{1}{1+\nu}\frac{\partial^2 s}{\partial y^2} = -2\frac{\partial f_y}{\partial y} - \frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) \\ \nabla^2 \sigma_z + \frac{1}{1+\nu}\frac{\partial^2 s}{\partial z^2} = -2\frac{\partial f_z}{\partial z} - \frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) \end{aligned}\right\}$$

$$\left.\begin{aligned} \nabla^2 \tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2 s}{\partial x \partial y} = -\left(\frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial x}\right) \\ \nabla^2 \tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2 s}{\partial y \partial z} = -\left(\frac{\partial f_y}{\partial z} + \frac{\partial f_z}{\partial y}\right) \\ \nabla^2 \tau_{xz} + \frac{1}{1+\nu}\frac{\partial^2 s}{\partial x \partial z} = -\left(\frac{\partial f_x}{\partial z} + \frac{\partial f_z}{\partial x}\right) \end{aligned}\right\}$$

Ecuaciones cinemáticas: $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$, $\omega_{ij} = \frac{1}{2}\left(u_{j,i} - u_{i,j}\right)$

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \varepsilon_x = \frac{\partial w}{\partial z};\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\omega_{xy} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \omega_{xz} = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \quad \omega_{yz} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

Ecuaciones constitutivas:

Hooke: $\varepsilon_{ij} = \frac{\sigma_{ij}}{2G} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \quad \gamma_{xz} = \frac{\tau_{xz}}{G} \quad \gamma_{yz} = \frac{\tau_{yz}}{G}$$

Lamé: $\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} = \frac{E}{\left(1+\nu\right)\left(1-2\nu\right)}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}$$

$$\tau_{xy} = G\gamma_{xy} \quad \tau_{xz} = G\gamma_{xz} \quad \tau_{yz} = G\gamma_{yz}$$

Parámetros varios: $\lambda = \frac{\nu E}{\left(1+\nu\right)\left(1-2\nu\right)};\ G = \frac{E}{2\left(1+\nu\right)};\ K = \frac{E}{3\left(1-2\nu\right)}$

$$s = \sigma_{kk} = \sigma_x + \sigma_y + \sigma_z;\ e = \varepsilon_{kk} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

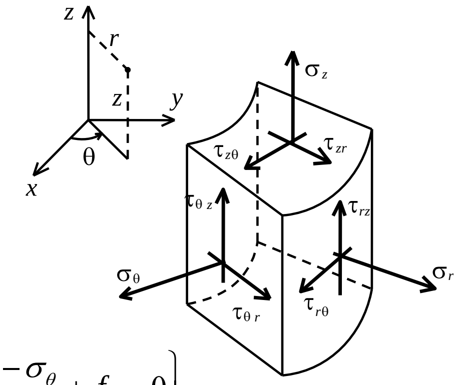
$$s = \left(\lambda + \frac{2}{3}G\right)e;\ s = Ke$$

Trabajo fuerzas exteriores: $W = \frac{1}{2}\int_{Vol}\left(f_x u + f_y v + f_z w\right)dVol + \frac{1}{2}\int_A\left(\overline{f}_x u + \overline{f}_y v + \overline{f}_z w\right)dA$

Energía elástica:

$$U = \int_{Vol}\mathbf{T}:\mathbf{D}dVol = \int_{Vol}\left[\frac{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}{2E} - \frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z\right) + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2\right)\right]dVol$$

ELASTICIDAD TRIDIMENSIONAL. COORDENADAS CILÍNDRICAS



Tensor de deformaciones: $\mathbf{D} = \begin{bmatrix} \varepsilon_r & \frac{1}{2}\gamma_{r\theta} & \frac{1}{2}\gamma_{rz} \\ \frac{1}{2}\gamma_{r\theta} & \varepsilon_\theta & \frac{1}{2}\gamma_{\theta z} \\ \frac{1}{2}\gamma_{rz} & \frac{1}{2}\gamma_{\theta z} & \varepsilon_z \end{bmatrix}$

Tensor de tensiones: $\mathbf{T} = \begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{r\theta} & \sigma_\theta & \tau_{\theta z} \\ \tau_{rz} & \tau_{\theta z} & \sigma_z \end{bmatrix}$

Equilibrio interno (tensiones):

$$\left.\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + f_z = 0 \end{aligned}\right\}$$

Compatibilidad interna (deformaciones):

$$\left.\begin{aligned} \frac{\partial \varepsilon_z}{\partial r} + r\frac{\partial^2 \varepsilon_\theta}{\partial z^2} - \frac{\partial \gamma_{rz}}{\partial z} = 0 \\ \varepsilon_r - \varepsilon_\theta - r\frac{\partial \varepsilon_\theta}{\partial r} = 0 \end{aligned}\right\}$$

Ecuaciones cinemáticas: $\varepsilon_r = \frac{\partial u_r}{\partial r} \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad \varepsilon_z = \frac{\partial u_z}{\partial z}$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \quad \gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta} \quad \gamma_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}$$

Ecuaciones constitutivas:

Hooke: $\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix}$ Lamé: $\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu \\ \nu & 1-\nu & \nu \\ \nu & \nu & 1-\nu \end{bmatrix}\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix}$

$\gamma_{r\theta} = \frac{\tau_{r\theta}}{G} \quad \gamma_{rz} = \frac{\tau_{rz}}{G} \quad \gamma_{\theta z} = \frac{\tau_{\theta z}}{G}$ $\tau_{r\theta} = G\gamma_{r\theta} \quad \tau_{rz} = G\gamma_{rz} \quad \tau_{\theta z} = G\gamma_{\theta z}$

Solución a partir de una función de tensiones:

Condición fundamental: $\nabla^2\nabla^2\Phi = 0$ $\nabla^2\Phi = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial \theta^2} + \frac{\partial^2\Phi}{\partial z^2}$

Obtención de tensiones (caso axil-simétrico):

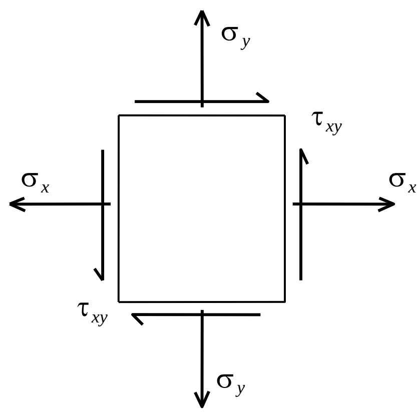
$$\left.\begin{aligned} \sigma_r &= \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right) \\ \sigma_\theta &= \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{1}{r}\frac{\partial \Phi}{\partial r}\right) \\ \sigma_z &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right] \\ \tau_{rz} &= \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right] \end{aligned}\right\}$$

ELASTICIDAD BIDIMENSIONAL. COORDENADAS CARTESIANAS

TENSIÓN PLANA (laja o placa de pequeño espesor)

Tensor de deformaciones: $\mathbf{D}=\begin{bmatrix}\varepsilon_x & \frac{1}{2}\gamma_{xy} & 0\\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0\\ 0 & 0 & \varepsilon_z\end{bmatrix}$

Tensor de tensiones: $\mathbf{T}=\begin{bmatrix}\sigma_x & \tau_{xy} & 0\\ \tau_{xy} & \sigma_y & 0\\ 0 & 0 & 0\end{bmatrix}$



Equilibrio interno (tensiones):

$$\left.\begin{array}{l}\dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}+f_x=0\\ \dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+f_y=0\\ f_z=0\end{array}\right\}$$

Equilibrio interno (movimientos):

$$\left.\begin{array}{l}G\nabla^2u_x+G\dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial x}+f_x=0\\ G\nabla^2u_y+G\dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial y}+f_y=0\\ e=\dfrac{1-2\nu}{1-\nu}\left(\varepsilon_x+\varepsilon_y\right)\end{array}\right\}$$

Compatibilidad en deformaciones:

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

Compatibilidad en tensiones:

$$\nabla^2\left(\sigma_x+\sigma_y\right)=-\left(1+\nu\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)$$

Ecuaciones constitutivas:

Hooke: $\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}=\dfrac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2\left(1+\nu\right)\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$

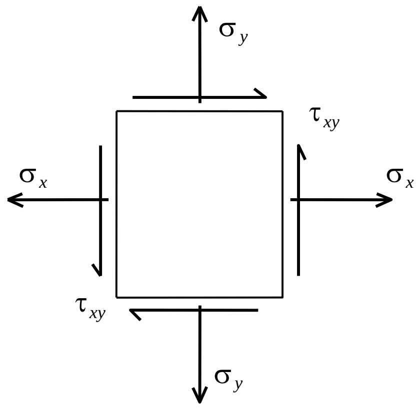
$$\varepsilon_z=-\frac{\nu}{E}\left(\sigma_x+\sigma_y\right)$$

Lamé: $\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}=\dfrac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$

ELASTICIDAD BIDIMENSIONAL. COORDENADAS CARTESIANAS

DEFORMACIÓN PLANA (barra o tubo de gran longitud)

Tensor de deformaciones: $\mathbf{D} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$



Tensor de tensiones: $\mathbf{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$

Equilibrio interno (tensiones):

$$\left.\begin{array}{c} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{xy}}{\partial y} + f_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + f_y = 0 \\ f_z = 0 \end{array}\right\}$$

Equilibrio interno (movimientos):

$$\left.\begin{array}{c} G\nabla^2 u_x + G\dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial x} + f_x = 0 \\ G\nabla^2 u_y + G\dfrac{1}{1-2\nu}\dfrac{\partial e}{\partial y} + f_y = 0 \\ e = \varepsilon_x + \varepsilon_y \end{array}\right\}$$

Compatibilidad en deformaciones:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Compatibilidad en tensiones:

$$\nabla^2\left(\sigma_x + \sigma_y\right) = -\frac{1}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

Ecuaciones constitutivas:

Hooke: $$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{1+\nu}{E}\begin{bmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

Lamé: $$\left.\begin{array}{c} \sigma_x = \lambda e + 2G\varepsilon_x \\ \sigma_y = \lambda e + 2G\varepsilon_y \end{array}\right\}$$

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

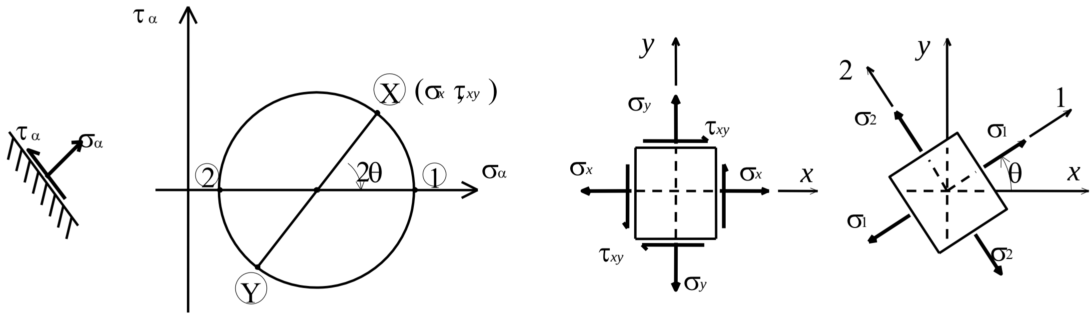
ELASTICIDAD BIDIMENSIONAL. COORDENADAS CARTESIANAS

SOLUCIÓN A PARTIR DE UNA FUNCIÓN DE TENSIONES
(fuerzas de masa constantes)

Condición fundamental de la función de Airy: $\nabla^4\Phi = 0 \qquad \dfrac{\partial^4\Phi}{\partial x^4} + 2\dfrac{\partial^4\Phi}{\partial x^2 \partial y^2} + \dfrac{\partial^4\Phi}{\partial y^4} = 0$

Obtención de tensiones:
$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\Phi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2\Phi}{\partial x^2} \\ \tau_{xy} &= -\frac{\partial^2\Phi}{\partial x \partial y} - x f_y - y f_x \end{aligned}\right\}$$

LÍNEAS CARACTERÍSTICAS



Tensiones principales: $\sigma_{1,2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Tensión tangencial máxima: $\tau_{\max} = \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

Tensiones alrededor de un punto:
$$\begin{aligned} \sigma_\alpha &= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy} \sin\alpha \cos\alpha \\ \tau_\alpha &= \left(\sigma_y - \sigma_x\right)\sin\alpha\cos\alpha + \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right) \end{aligned}$$

Ángulo de las tensiones principales con el eje x: $\tan 2\theta = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$; $\tan\theta_i = \dfrac{\sigma_i - \sigma_x}{\tau_{xy}}$, $i = 1,2$

Líneas isostáticas: $\dfrac{dy}{dx} = -\dfrac{\sigma_x - \sigma_y}{2\tau_{xy}} \pm \sqrt{1 + \left(\dfrac{\sigma_x - \sigma_y}{2\tau_{xy}}\right)^2}$

Líneas isoclinas: $\tan 2\varphi = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y} = cte$

Líneas isobaras: $\sigma_{1,2} = \dfrac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = cte$

Líneas de máxima tensión tangencial: $\dfrac{dy}{dx} = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y} \pm \sqrt{1 + \left(\dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}\right)^2}$

Deformaciones principales: $\varepsilon_{1,2} = \dfrac{1}{2}\left[\varepsilon_x + \varepsilon_y \pm \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + 4\gamma_{xy}^2}\right]$

Deformación tangencial máxima: $\gamma_{\max} = \sqrt{\left(\varepsilon_x - \varepsilon_y\right)^2 + 4\gamma_{xy}^2}$

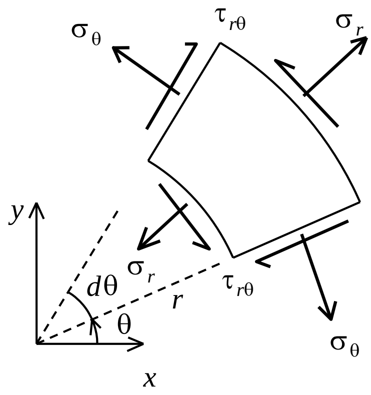
Deformaciones alrededor de un punto:

$$\varepsilon_\alpha = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$\gamma_\alpha = \left(\varepsilon_x - \varepsilon_y\right)\sin 2\alpha - \gamma_{xy}\cos 2\alpha$$

Ángulo de las deformaciones principales con el eje *x*: $\tan 2\theta = \dfrac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$, $\tan\theta_i = \dfrac{\varepsilon_i - \varepsilon_x}{\varepsilon_{xy}}$

ELASTICIDAD BIDIMENSIONAL. COORDENADAS POLARES



Tensor de deformaciones: $\mathbf{D}=\begin{bmatrix}\varepsilon_r & \frac{1}{2}\gamma_{r\theta}\\ \frac{1}{2}\gamma_{r\theta} & \varepsilon_\theta\end{bmatrix}$

Tensor de tensiones: $\mathbf{T}=\begin{bmatrix}\sigma_r & \tau_{r\theta}\\ \tau_{r\theta} & \sigma_\theta\end{bmatrix}$

Equilibrio interno (tensiones): $$\left.\begin{array}{l}\dfrac{\partial\sigma_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial\tau_{r\theta}}{\partial\theta}+\dfrac{\sigma_r-\sigma_\theta}{r}+f_r=0\\ \dfrac{1}{r}\dfrac{\partial\sigma_\theta}{\partial\theta}+\dfrac{\partial\tau_{r\theta}}{\partial r}+2\dfrac{\tau_{r\theta}}{r}+f_\theta=0\end{array}\right\}$$

Compatibilidad interna (tensiones): $\nabla^2\left(\sigma_r+\sigma_\theta\right)=0$

Ecuaciones constitutivas:

Tensión plana:

$$\begin{bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu\\ -\nu & 1\end{bmatrix}\begin{bmatrix}\sigma_r\\ \sigma_\theta\end{bmatrix};\ \gamma_{r\theta}=\frac{\tau_{r\theta}}{G}$$

$$\begin{bmatrix}\sigma_r\\ \sigma_\theta\end{bmatrix}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}\begin{bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{bmatrix};\ \tau_{r\theta}=G\gamma_{r\theta}$$

$$\varepsilon_z=-\frac{\nu}{E}\left(\sigma_r+\sigma_\theta\right)$$

Deformación plana:

$$\begin{bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{bmatrix}=\frac{1+\nu}{E}\begin{bmatrix}1-\nu & -\nu\\ -\nu & 1-\nu\end{bmatrix}\begin{bmatrix}\sigma_r\\ \sigma_\theta\end{bmatrix};\ \gamma_{r\theta}=\frac{\tau_{r\theta}}{G}$$

$$\begin{bmatrix}\sigma_r\\ \sigma_\theta\end{bmatrix}=\frac{E}{\left(1+\nu\right)\left(1-2\nu\right)}\begin{bmatrix}1-\nu & \nu\\ \nu & 1-\nu\end{bmatrix}\begin{bmatrix}\varepsilon_r\\ \varepsilon_\theta\end{bmatrix};\ \tau_{r\theta}=G\gamma_{r\theta}$$

$$\sigma_z=\nu\left(\sigma_r+\sigma_\theta\right)$$

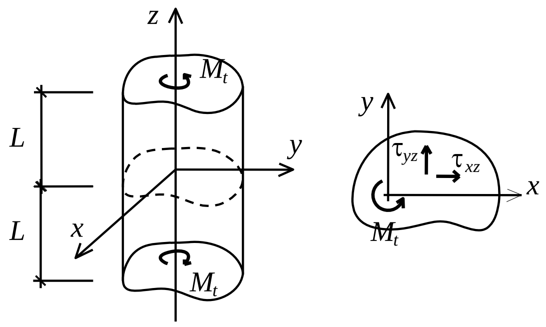
Ecuaciones cinemáticas: $\varepsilon_r=\dfrac{\partial u_r}{\partial r}$ $\quad\varepsilon_\theta=\dfrac{u_r}{r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}$ $\quad\gamma_{r\theta}=\dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r}$

Solución a partir de una función de tensiones:

Condición fundamental: $\nabla^2\nabla^2\Phi=0$ $\qquad\nabla^2\Phi=\dfrac{\partial^2\Phi}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial\Phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\Phi}{\partial\theta^2}$

Obtención de tensiones: $$\left.\begin{array}{l}\sigma_r=\dfrac{1}{r}\dfrac{\partial\Phi}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2\Phi}{\partial\theta^2}\\ \sigma_\theta=\dfrac{\partial^2\Phi}{\partial r^2}\\ \tau_{r\theta}=-\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\dfrac{\partial\Phi}{\partial\theta}\right)\end{array}\right\}$$

TORSIÓN UNIFORME



SOLUCIÓN EN MOVIMIENTOS (SAINT VENANT)

Ángulo girado por unidad de longitud: $\vartheta = \dfrac{\omega}{L}$

Ángulo girado en una sección cualquiera: $\omega(z) = \vartheta z$

Movimientos: $\left.\begin{array}{r} u = -\vartheta yz \\ v = \vartheta xz \\ w = \vartheta f(x,y) \end{array}\right\}$, siendo $f(x,y)$ la función de alabeo, tal que $\nabla^2 f = 0$

Ecuación constitutiva: $M_t = GJ\vartheta$ $J = \int_A \left(\dfrac{\partial f}{\partial y} x - \dfrac{\partial f}{\partial x} y + x^2 + y^2 \right) dxdy$

SOLUCIÓN EN TENSIONES (PRANDTL)

Solución a partir de una función de tensiones:

Condición fundamental (compatibilidad): $\nabla^2 \Phi = cte = -2G\vartheta$ $\left.\dfrac{\partial \Phi}{\partial s}\right|_{contorno} = 0$

$\Phi\big|_{contorno} = cte$

Obtención de tensiones: $\left.\begin{array}{r} \tau_{xz} = \dfrac{\partial \Phi}{\partial y} \\ \tau_{yz} = -\dfrac{\partial \Phi}{\partial x} \\ \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \end{array}\right\}$

Equilibrio en las secciones extremas: $M_t = 2\int_A \Phi dxdy$

Energía potencial total: $V = U - W = \dfrac{1}{2G} \int_A \left[\left(\dfrac{\partial \Phi}{\partial x} \right)^2 + \left(\dfrac{\partial \Phi}{\partial y} \right)^2 - 4G\vartheta\Phi \right] dxdy$

FÓRMULAS PRÁCTICAS PARA ALGUNAS SECCIONES

Circular de radio *R* (*f*(*x*,*y*) = 0): $\tau_{xz} = -\frac{2M_t}{\pi R^4} y \quad \tau_{yz} = \frac{2M_t}{\pi R^4} x \quad \tau_{max} = \frac{2M_t}{\pi R^3} \quad \frac{\omega}{L} = \frac{2M_t}{G\pi R^4}$

Cuadrada de lado *a*: $\tau_{max} = \frac{M_t}{0.208a^3} \quad M_t = 0.1404\frac{G\omega}{L}a^4$

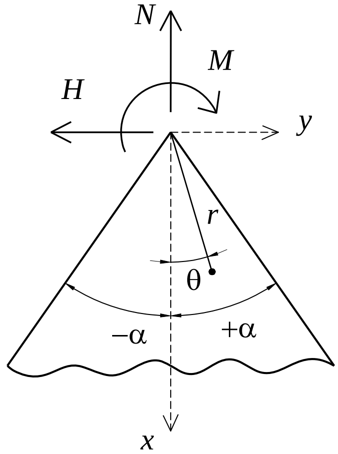
Rectangular de lados *a* y *b*:

$$\tau_{max} = \frac{M_t}{\left[0.33 - 0.22\frac{a}{b} + 0.10\left(\frac{a}{b}\right)^2\right]a^2 b}$$

$$M_t = \frac{G\omega}{L}a^3 b\left[0.33 - 0.22\frac{a}{b} + 0.10\left(\frac{a}{b}\right)^2\right]$$

SOLUCIONES A ALGUNOS PROBLEMAS ELÁSTICOS

Voladizo de caras oblicuas:

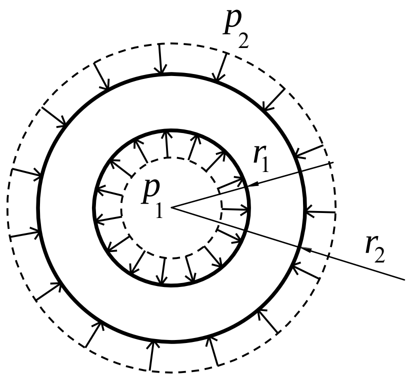


$$\sigma_r = \frac{N\cos\theta}{\left(\alpha + \frac{1}{2}\sin 2\alpha\right) r} + \frac{H\sin\theta}{\left(\alpha - \frac{1}{2}\sin 2\alpha\right) r} - \frac{M\sin 2\theta}{\left(\frac{1}{2}\sin 2\alpha - \alpha\cos 2\alpha\right) r^2}$$

$$\sigma_\theta = 0$$

$$\tau_{r\theta} = \frac{M\left(\cos 2\theta - \cos 2\alpha\right)}{\left(\sin 2\alpha - 2\alpha\cos 2\alpha\right) r^2}$$

Tubo circular sometido a presiones radiales:



$$\sigma_r = \frac{A}{r^2} + 2C$$

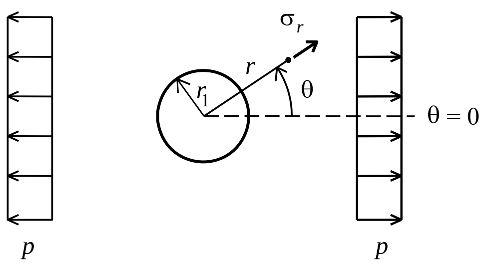
$$\sigma_\theta = -\frac{A}{r^2} + 2C$$

$$\tau_{r\theta} = 0$$

$$\sigma_z = \nu\left(\sigma_r + \sigma_\theta\right) \quad \tau_{rz} = \tau_{r\theta} = 0 \quad \varepsilon_z = cte$$

$$A = -\frac{r_1^2 r_2^2\left(p_1 - p_2\right)}{r_2^2 - r_1^2} \quad 2C = \frac{p_1 r_1^2 - p_2 r_2^2}{r_2^2 - r_1^2}$$

Taladro circular en una chapa indefinida:



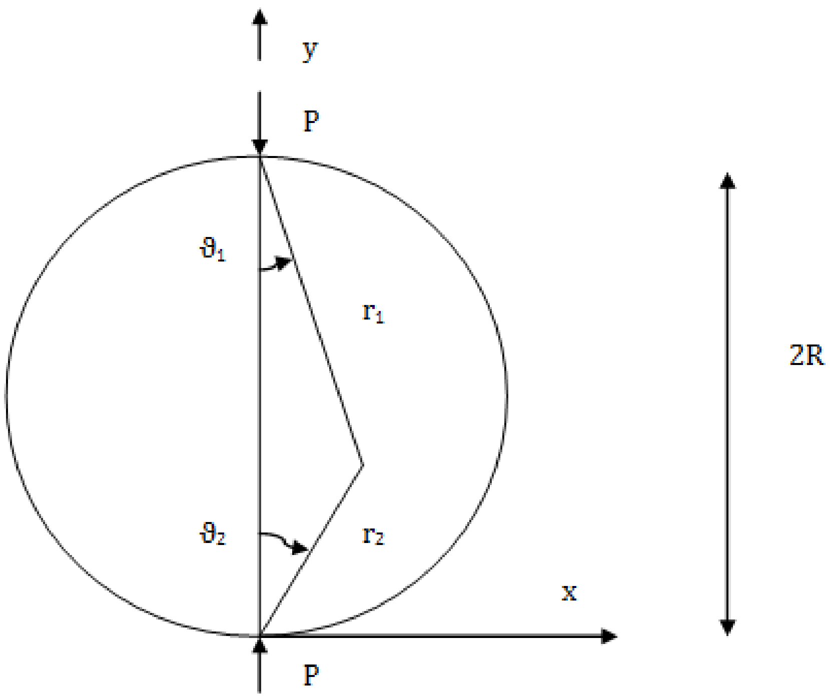
$$\sigma_r = \frac{p}{2} - \frac{p}{2}\xi^2 + \frac{p}{2}\left(1 + 3\xi^4 - 4\xi^2\right)\cos 2\theta$$

$$\sigma_\theta = \frac{p}{2} + \frac{p}{2}\xi^2 - \frac{p}{2}\left(1 + 3\xi^4\right)\cos 2\theta$$

$$\tau_{r\theta} = -\frac{p}{2}\left(1 - 3\xi^4 + 2\xi^2\right)\sin 2\theta$$

siendo $\xi = \frac{r_1}{r}$

Cilindro sometido a dos cargas a lo largo de generatrices opuestas



$$\sigma_x = -\frac{2P}{\pi}\left[\frac{\cos\vartheta_1 sen^2\vartheta_1}{r_1} + \frac{\cos\vartheta_2 sen^2\vartheta_2}{r_2} - \frac{1}{2R}\right]$$

$$\sigma_y = -\frac{2P}{\pi}\left[\frac{\cos^3\vartheta_1}{r_1} + \frac{\cos^3\vartheta_2}{r_2} - \frac{1}{2R}\right]$$

$$\tau_{xy} = -\frac{2P}{\pi}\left[\frac{\cos^2\vartheta_2 sen\vartheta_2}{r_2} - \frac{\cos^2\vartheta_1 sen\vartheta_1}{r_1}\right]$$